

# Extremal Black Holes and Holographic C-Theorem

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## Abstract

We found Bogomol'nyi type of the first order differential equations in three dimensional Einstein gravity and the effective second order ones in new massive gravity when an interacting scalar field is minimally coupled. Using these equations in Einstein gravity, we obtain analytic solutions corresponding to extremally rotating hairy black holes. We also obtain perturbatively extremal black hole solutions in new massive gravity using these lower order differential equations. All these solutions have the anti de-Sitter spaces as their asymptotic geometries and as the near horizon ones. This feature of solutions interpolating two anti de-Sitter spaces leads to the construction of holographic c-theorem in these cases. Since our lower order equations reduce naturally to the well-known equations for domain walls, our results can be regarded as the natural extension of domain walls to more generic cases.

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# 1 Introduction

Recently, there has been much interest in  $c$ -theorem in various dimensions. The content of the theorem is that a certain central charge function, which is regarded as counting the number of degrees of freedom, should be a monotonically decreasing function along a Wilsonian renormalization group (RG) flow. The interesting recent developments include the formal ‘proof’ [1] of four-dimensional  $c$ -(or  $a$ -)theorem long after its conjecture [2], the discovery of the relation between central charge and the entanglement entropy (EE) [3], and the identification of the conjectured free energy maximization in three-dimensional field theory named as F-maximization with the internal space volume minimization known as Z-minimization [4]. Furthermore, the content of this theorem was constructed holographically in Einstein gravity through the AdS/CFT correspondence [5, 6] and extended recently to higher derivative gravity [7]. The pioneering work on all these developments in  $c$ -theorem was done by Zamolodchikov a few decades ago [8], who argued that  $c$ -theorem is natural intuitively in the sense of RG flow from UV to IR and proved rigorously that it holds in two dimensions under the assumption on generic properties of field theory like unitarity, conformal invariance, etc.

In general, it seems natural that a kind of  $c$ -theorem holds in any kind of sensible unitary field theories. Contrary to this simple intuition, it is challenging to prove this theorem because its nature requires essentially the non-perturbative method and furthermore its proof or verification depends strongly on the spacetime dimensions, in the works that have been done so far. There have been some attempts to overcome this situation. For instance, the relation between central charge and EE was shown in two dimensions, and then two-dimensional  $c$ -theorem is rederived by using the property of EE. This construction is extended to higher dimensions and argued to be a generic proof of  $c$ -theorem. Other attempts are the holographic construction of  $c$ -theorem in various gravity models, which reveal that the conjectured  $c$ -theorem is consistent with the holographic construction. In four dimensions there are two central charges called as  $a$  and  $c$ , which are the coefficients of Euler and Weyl density in the trace anomaly formula. It was conjectured and proved recently that  $a$  is the relevant central charge consistent with the monotonic flow property. The holographic construction is very appealing since it reproduces this result nicely and it can be extended to various other dimensions including odd ones.

Though the holographic construction is very useful to understand  $c$ -theorem uniformly in various dimensions, its construction is restricted usually to the simplest form of domain wall metric in the gravity side. In the context of the AdS/CFT correspondence, domain wall solutions in gravity with two asymptotic AdS spaces correspond to RG flow trajectory between two conformal points in dual field theory. In this holographic construction, appropriate central charge flow functions, which coincide with central charges at the two conformal points, are constructed by using metric functions. And then their monotonicity is verified through the equations of motion and null energy condition on matters which is imposed as a sensible condition in the gravity side. This is the content of the so-called holographic  $c$ -theorem.

Even though this holographic  $c$ -theorem may be checked without the explicit domain wall

solutions, it is more satisfactory to obtain the analytic domain wall solutions consistent with holographic  $c$ -theorem. Interestingly, it has been shown that the domain wall metric in Einstein gravity satisfies Bogomol'nyi type of the first order differential equations which were derived by minimizing a certain energy functional through complete squares [9] or by introducing a certain 'fake' supersymmetry or supergravity [10, 11]. Using these first order equations, various analytic forms of domain wall solutions have been obtained and shown to be consistent with holographic  $c$ -theorem in various dimensions.

One may ask whether this domain wall geometry is the unique candidate as the dual to the RG flow in boundary field theory. It is rather clear that any geometry with two asymptotic AdS spaces is viable as the dual to the RG flow and as the background for holographic  $c$ -theorem. However, it is not easy to obtain such non-trivial background geometry analytically in gravity since matters play important roles and hinder analytic treatments. In this regard, three-dimensional gravity is exceptional since various analytic solutions are found including black holes with scalar hairs [12]. Indeed, there is the realization of this idea by using more complicated three-dimensional geometry than domain walls, which turns out to be consistent with holographic  $c$ -theorem [13]. The relevant geometry is given by the extremally rotating three-dimensional AdS black holes which interpolate between AdS space at the asymptotic infinity and near horizon AdS geometry.

The existence of the analytic black hole solutions allows the explicit realization of holographic RG flow via Hamilton-Jacobi formalism and the check of the holographic  $c$ -theorem. However, the shortcoming in these extremal black hole solutions given in Ref. [13], compared to the domain wall solutions, is the fact that one needs to make a specific 'ad hoc' choice of the scalar potential to obtain analytic results. This point is even amplified when one considers higher curvature gravity like new massive gravity(NMG) [14] which is recently introduced as a non-linear completion of Pauli-Fierz linear massive graviton theory and shown to be consistent with a simple form of a holographic  $c$ -theorem [15]. It becomes very difficult to choose 'ad hoc' scalar potential in the NMG case, which is contrasted to the fact that domain walls satisfy first order differential equations even in NMG and allow analytic results [16].

In the context of the AdS/CFT correspondence, the generic nature of holographic construction seems to imply that there exists a more unified and systematic approach to these extremal black holes with two AdS asymptotics. It is natural to suspect the existence of some reduced differential equations for extremal black holes as for domain walls. One of main results in this paper is the discovery of such differential equations for three-dimensional AdS black holes in Einstein gravity and in NMG. We also show that such equations are enough for the consistency with the holographic  $c$ -theorem when a certain central charge flow function is chosen.

This paper is organized as follows. In the next section we find the Bogomol'nyi type of first order differential equations, which solves the full equations of motion, in three-dimensional Einstein gravity interacting with a scalar field. It turns out that these restricted first order equations of motion represent extremally rotating black holes with scalar hairs. By solving these first order equations of motion in a more or less systematic way, we obtain some analytic

hairy black hole solutions which include the case given in Ref. [13] as a special case. In section three we consider new massive gravity as another gravity theory to obtain reduced differential equations for extremal AdS black holes. As in the Einstein gravity case, it is shown that Bogomol'nyi type of lower order differential equations can be obtained which include domain wall solutions as special cases. By solving these equations asymptotically, we show that there are extremally rotating black hole solutions consistent with a holographic  $c$ -theorem. In section four, we consider the holographic  $c$ -theorem in our setup and show that it holds generically by using reduced lower order equations of motions. In the final section, we summarize our results with some comments and discuss open issues.

## 2 Extremal Black Hole Solutions in Einstein Gravity

In this section we consider three-dimensional Einstein gravity with a minimally coupled interacting scalar field. We find Bogomol'nyi type of first order differential equations which solve full equations of motion. This can be regarded as the extension of first order equations for domain walls [9, 10] to more generic cases. It turns out that the simplest solutions of these equations, which are given by a constant scalar field, correspond to extremal BTZ black holes [17]. After showing that these equations describe the extremally rotating black holes, we obtain analytic solutions of some hairy black holes in a systematic way.

### 2.1 First order equations of motion

In the convention of mostly plus signs for the metric with the convention of curvature tensors as  $[\nabla_\mu \nabla_\nu]V_\rho = R_{\mu\nu\rho\sigma}V^\sigma$  and  $R_{\mu\nu} = g^{\alpha\beta}R_{\alpha\mu\beta\nu}$ , our starting action for Einstein gravity with a minimally coupled scalar field is given by

$$S = \frac{1}{16\pi G} \int d^3x \sqrt{-g} \left[ R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right], \quad (1)$$

of which the equations of motions(EOM) are composed of scalar field equation and the metric field equations as follows ;

$$0 = E_\phi \equiv \nabla^2 \phi - \frac{\partial V}{\partial \phi}, \quad 0 = E_{\mu\nu} \equiv \mathcal{E}_{\mu\nu} - T_{\mu\nu}, \quad (2)$$

where

$$\mathcal{E}_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}, \quad T_{\mu\nu} \equiv \frac{1}{2} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \left[ \frac{1}{2} \partial_\alpha \phi \partial^\alpha \phi + V(\phi) \right].$$

To find asymptotically AdS black hole solutions with an interacting scalar field in three dimensions, let us take our metric ansatz in AdS-Schwarzschild-like coordinates as

$$ds^2 = L^2 \left[ -e^{2A(r)} dt^2 + e^{2B(r)} dr^2 + r^2 \left( d\theta + e^{C(r)} dt \right)^2 \right], \quad (3)$$

where  $L$  denotes the radius of asymptotic AdS space. Asymptotically AdS black holes in these coordinates mean that the asymptotic conditions on the functions  $A(r), B(r), C(r)$  are given as follows;

$$e^{A(r)} \Big|_{r \rightarrow \infty} \rightarrow r, \quad e^{B(r)} \Big|_{r \rightarrow \infty} \rightarrow \frac{1}{r}, \quad e^{C(r)} \Big|_{r \rightarrow \infty} \rightarrow \text{const.} + \mathcal{O}\left(\frac{1}{r^2}\right). \quad (4)$$

Note that these boundary conditions are Brown-Henneaux type which allow us to apply the standard central charge extraction by Brown-Henneaux method [18].

The equations of motion even in this case turn out to be complicated non-linear differential equations. For instance, the EOM for the scalar field is given by

$$0 = E_\phi \equiv \frac{1}{L^2} e^{-2B} \left[ \left( A' - B' + \frac{1}{r} \right) \phi' + \phi'' \right] - \frac{\partial V}{\partial \phi}, \quad (5)$$

where  $'$  denotes differentiation with respect to the radial coordinate  $r$ . The EOM for metric,  $0 = E_{\mu\nu}$ , are relegated to appendix A. To obtain analytic solutions of complicated full EOM, it is very convenient to introduce the so-called ‘superpotential’ method which is originally applied to the domain wall solutions. Historically, the terminology of superpotential is chosen in analogy with supergravity expression for a scalar potential. When the scalar potential is represented by the so-called superpotential  $\mathcal{W}$  as

$$V(\phi) = \frac{1}{2L^2} \left( \frac{\partial \mathcal{W}}{\partial \phi} \right)^2 - \frac{1}{2L^2} \mathcal{W}^2, \quad (6)$$

the appropriate first order differential equations, which solve full EOM, are given by

$$\begin{aligned} \phi' &= -e^B \frac{\partial \mathcal{W}}{\partial \phi}, & A' &= e^B \mathcal{W} - \frac{1}{r}, \\ A' + B' &= \frac{r}{2} e^{2B} \left( \frac{\partial \mathcal{W}}{\partial \phi} \right)^2 = -\frac{r}{2} e^B \left( \frac{\partial \mathcal{W}}{\partial \phi} \right) \phi' = \frac{r}{2} \phi'^2, \\ (e^C)' &= \mp \frac{1}{r} e^A \left( e^B \mathcal{W} - \frac{2}{r} \right) = \mp \left( \frac{1}{r} e^A \right)'. \end{aligned} \quad (7)$$

These are motivated by similar expression in the domain wall case. This form of differential equations, which we call as reduced EOM, is a considerable simplification compared to the original EOM, though restricted solutions among all possible ones can be obtained from these reduced EOM. Specifically, the last equation can be solved as

$$e^C = C_\mp \mp \frac{1}{r} e^A, \quad (8)$$

where the integration constants  $C_\mp$  can take any values consistently with asymptotic AdS space.

As a trivial solution of our first order equations, let us consider the constant potential case,  $V = -2/L^2$  with a constant scalar field. In this case superpotential is given by  $\mathcal{W} = 2$  and then one obtains

$$A' = 2e^B - \frac{1}{r}, \quad A' + B' = 0, \quad (9)$$

which, with boundary conditions, leads to the following solution;

$$e^A = e^{-B} = r - \frac{r_H^2}{r}, \quad e^C = (C_{\mp} \mp 1) \pm \frac{r_H^2}{r^2}. \quad (10)$$

These metric functions represent the extremal BTZ black holes. Although  $C_{\mp} = \pm 1$  corresponds to the most familiar form of extremal BTZ black holes, any value of the constant,  $C_{\mp}$ , leads to the extremal BTZ black holes. In fact, it is more useful to take  $C_{\mp} = 0$  to simplify some computations in our case<sup>1</sup>. To see the convenience of this choice, let us consider the near horizon geometry of extremal BTZ black holes given by  $r \rightarrow r_H$ . Using a new radial coordinate  $\rho = 4(r - r_H)$ , one can easily identify the metric of the near horizon geometry as

$$ds_{NH}^2 = \frac{L^2}{4} \left[ -\rho^2 dt^2 + \frac{1}{\rho^2} d\rho^2 + 4r_H^2 \left( d\theta \mp \frac{\rho}{2r_H} dt \right)^2 \right],$$

which is the well-known metric form [19] of the self-dual orbifold of  $AdS_3$  with the radius  $L$ . Note that this geometry leads to zero Hawking temperature and so dual field theory to extremal BTZ can be thought to be at zero temperature. This near horizon geometry is interpreted as dual to the discrete light cone quantization(DLCQ) of two-dimensional conformal field theory(CFT) and is related to chiral two-dimensional CFT [20].

From now on we will choose the integration constant as  $C_{\mp} = 0$  so that  $C$  is given by

$$e^C = \mp \frac{1}{r} e^A. \quad (11)$$

The usual choice of  $C_{\mp} = \pm 1$  can be recovered by a simple coordinate transformation:  $\theta \rightarrow \theta + C_{\mp} t$ .

One may expect that all the solutions of our first order equations correspond to some kind of extremal black holes as can be inferred by the fact that the trivial solutions represent the extremal BTZ black holes. This expectation is also natural in analogy with charged extremal black hole solutions in supergravity, in which those black holes are described by first order equations which can be derived by Killing spinor equations. As in the case of domain walls, we anticipate that some ‘fake’ Killing spinor equations might lead to our first order equations. In the next section we present perturbative analysis of these first order equations and show that black hole solutions are indeed extremal.

## 2.2 Extremally rotating black holes

It is very convenient in solving the first order reduced EOM to take  $\phi$  as coordinates and  $r$  as a function of  $\phi$ , instead of the original form. Then the first order reduced EOM can be rewritten

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<sup>1</sup>However, when we consider conserved charges of black holes, which depend on the coordinates, we return to the coordinates  $C_{\mp} = \pm 1$  to reproduce the standard form of conserved charges.

as

$$\begin{aligned}
\partial_\phi(A+B)\partial_\phi(\ln r) &= \frac{1}{2}, \\
e^B \partial_\phi r \partial_\phi \mathcal{W} &= -1, \\
\partial_\phi(A+\ln r)\partial_\phi(\ln \mathcal{W}) &= -1,
\end{aligned} \tag{12}$$

where  $\partial_\phi$  denotes differentiation with respect to  $\phi$  variable. One can see that the first two (or last two) equations can be immediately integrated in terms of  $r$  and  $\mathcal{W}$  and lead to solutions of metric functions  $A$  and  $B$  as

$$\begin{aligned}
A &= \frac{1}{2} \int^\phi d\phi' \left[ \partial_{\phi'} \ln r(\phi') \right]^{-1} + \ln \left[ -\partial_\phi r \partial_\phi \mathcal{W} \right] = -\ln r + \int^\phi d\phi' e^B \mathcal{W} \partial_{\phi'} r, \\
B &= -\ln \left[ -\partial_\phi r \partial_\phi \mathcal{W} \right].
\end{aligned}$$

By inserting the expression of  $A$  and  $B$  functions in the remaining equation, one obtains the differential equation for  $r(\phi)$  or for  $\mathcal{W}(\phi)$  as

$$0 = \left[ \partial_\phi^2 \mathcal{W} + \mathcal{W} \right] \partial_\phi \ln r + \left[ 2(\partial_\phi \ln r)^2 + \partial_\phi^2 \ln r + \frac{1}{2} \right] \partial_\phi \mathcal{W} = \frac{1}{2r^2} \partial_\phi \left[ r^2 \mathcal{W} + (\partial_\phi r^2)(\partial_\phi \mathcal{W}) \right], \tag{13}$$

which can be integrated as

$$r^2 \mathcal{W} + (\partial_\phi r^2)(\partial_\phi \mathcal{W}) = \text{constant} \equiv \Delta_0. \tag{14}$$

The physical meaning of this constant  $\Delta_0$  will be given shortly after discussing the near horizon geometry of our black hole solutions, which will also be related to the conserved charges. One can see that the metric functions  $A$  and  $B$  are now completely determined, in terms of the superpotential  $\mathcal{W}$  and the constant  $\Delta_0$ , as

$$A = \ln r - \Delta_0 \int^\phi d\phi' e^{B(\phi')} \partial_{\phi'} \left( \frac{1}{r} \right), \quad e^{-B} = \frac{r}{2} \left[ \mathcal{W} - \frac{1}{r^2} \Delta_0 \right]. \tag{15}$$

This form of the metric function  $B$  shows us explicitly that  $e^{-B}$  outside the horizon is a regular function of  $(r - r_H)$  as long as  $\mathcal{W}$  can be written as a regular function of  $(r - r_H)$ .

Before going ahead to present the analytic form of some hairy AdS black hole solutions, let us consider the asymptotic and near horizon behaviors of black hole solutions. According to the boundary conditions and physical consideration, one may take as

$$\begin{aligned}
A(r) &= \ln r + \frac{a_1}{r^2} + \dots, \\
B(r) &= -\ln r + \frac{b_1}{r^2} + \dots, \\
\ln \mathcal{W} &= \ln 2 + \omega_1 r^{-n} + \dots, \\
\phi(r) &= \phi_\infty + \frac{\phi_1}{r^k} + \dots,
\end{aligned} \tag{16}$$

where  $\phi_\infty$  denotes the value of scalar field  $\phi$  at the asymptotic infinity  $r = \infty$ . By solving the above differential equation (14) with (15) perturbatively, one can see that  $n = 2$ ,  $k = 1$  and obtain

$$\begin{aligned} a_1 &= -\frac{1}{2}\Delta_0, \quad b_1 = -\omega_1 + \frac{1}{2}\Delta_0, \\ \ln \mathcal{W} &= \ln 2 + \frac{1}{4}(\phi - \phi_\infty)^2 + \dots, \\ r^{-2} &= \frac{1}{4\omega_1}(\phi - \phi_\infty)^2 + \dots, \end{aligned} \tag{17}$$

where the last equation is obtained by inverting  $r$  as a function of  $\phi$  and using  $\phi_1^2 = 4\omega_1$ . Note that the superpotential value outside the horizon becomes always greater than its asymptotic value:  $\mathcal{W}(\phi) \geq \mathcal{W}(\phi_\infty)(= 2)$ . This fact can be checked explicitly from the analytic solutions presented in the following. Even in these perturbative solutions, the power of the first order reduced EOM shows up as the complete determination of  $a_2$  and  $b_2$  in terms of  $\omega_1$  and  $\Delta_0$ . Contrary to this, only the combination of  $a_1 + b_1 = -\omega_1$  is determined perturbatively by the original second order EOM.

Now, we would like to show that the first order reduced EOM represent extremal black holes by analyzing near horizon geometry of black hole solutions of the reduced EOM. By assuming the existence of the horizon, which is given by  $e^{-B(r_H)} = 0$ , with the expression of the metric function  $B$  in Eq. (15), one can see that the constant  $\Delta_0$  in Eq. (14) is determined as

$$\Delta_0 = r_H^2 \mathcal{W}(\phi_H). \tag{18}$$

By solving Eq. (14) perturbatively near the horizon, one can see that the superpotential  $\mathcal{W}$  and the radial coordinate can be taken as regular functions of  $\phi$  as

$$\begin{aligned} \mathcal{W}(\phi) &= \mathcal{W}(\phi_H) - \frac{1}{2}\mathcal{W}(\phi_H)(\phi - \phi_H)^2 + \dots, \\ r(\phi) &= r_H + h_0(\phi - \phi_H) + \dots, \end{aligned} \tag{19}$$

where  $h_0$  is a certain constant<sup>2</sup>. Accordingly, metric functions  $A$  and  $B$  are given by Eq. (15) as

$$\begin{aligned} e^{A(r)} &= s_0 \mathcal{W}(\phi_H)(r - r_H) + \dots, \\ e^{B(r)} &= \frac{1}{\mathcal{W}(\phi_H)(r - r_H)} + \dots, \end{aligned} \tag{20}$$

where  $s_0$  is a certain non-vanishing constant and  $\dots$  denotes some regular functions of  $(r - r_H)$ . Note that the constant  $s_0$  is related to the interval of integration in the expression of  $A$  given in Eq. (15). This form of the metric function  $B$  on the near horizon shows us the extremality of black hole solutions of our first order reduced EOM:

$$e^{-2B}\Big|_{r=r_H} = 0, \quad \frac{d}{dr}e^{-2B}\Big|_{r=r_H} = 0. \tag{21}$$

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<sup>2</sup>Here, we have assumed  $h_0 \neq 0$ . If this is not the case, the perturbative expression should be modified.



By introducing a new radial coordinate  $\rho \equiv s_0 \mathcal{W}^2(\phi_H)(r - r_H)$ , which is appropriate on the near horizon region, one may identify the near horizon geometry as

$$ds_{NH}^2 = \left[ \frac{L}{\mathcal{W}(\phi_H)} \right]^2 \left[ -\rho^2 dt^2 + \frac{1}{\rho^2} d\rho^2 + \hat{r}_H^2 \left( d\theta \mp \frac{\rho}{\hat{r}_H} dt \right)^2 \right], \quad \hat{r}_H \equiv r_H \mathcal{W}(\phi_H). \quad (22)$$

One may introduce the AdS scale  $\bar{L}$  on the near horizon geometry through  $V(\phi_H) = -2/\bar{L}^2$ , which is related to the superpotential as  $V(\phi_H) = -\mathcal{W}^2(\phi_H)/2L^2$  since  $\partial_\phi \mathcal{W}(\phi_H) = 0$  at the horizon. Interestingly, in terms of this scale  $\bar{L}$  and  $\bar{r}_H = \hat{r}_H/2$  the near horizon geometry may be written in the form of

$$ds_{NH}^2 = \frac{\bar{L}^2}{4} \left[ -\rho^2 dt^2 + \frac{1}{\rho^2} d\rho^2 + 4\bar{r}_H^2 \left( d\theta \mp \frac{\rho}{2\bar{r}_H} dt \right)^2 \right], \quad \bar{r}_H \equiv r_H \left[ \frac{L}{\bar{L}} \right], \quad (23)$$

which is just the metric for the self-dual orbifold of  $AdS_3$  with the radius  $\bar{L}$ . Now, one can identify  $\bar{L}$  with the superpotential value at the horizon  $\mathcal{W}(\phi_H)$  or the constant  $\Delta_0$  as

$$\bar{L} = \frac{2L}{\mathcal{W}(\phi_H)} = \frac{2r_H^2 L}{\Delta_0}. \quad (24)$$

This explains the physical meaning of the integration constant  $\Delta_0$ , which is related to the information about the near horizon geometry. This result will be consistent with the holographic  $c$ -theorem, as will be discussed in the section 4.

Now, let us consider some physical quantities related to these extremal black holes. One will see that  $\Delta_0$  is directly related to the conserved charges of black holes. Using the above explicit asymptotic expressions of the metric and the scalar field, one can obtain masses and angular momenta of black holes, for instance, through the so-called Abbott-Deser-Tekin(ADT) method [21]. Note that, in this ADT approach, one does not need to compute contributions separately from the metric and the scalar field, which is contrary to the quasi-local charge method given in Refs. [12, 13].

Since conserved charges depend on the coordinates, we need to specify those concretely. Here, we will choose those as  $C_\mp = \pm 1$  which gives us the standard metric form of BTZ black holes. That is to say, the background metric for ADT charge computation, which is  $AdS_3$  space, is taken in our coordinates as

$$ds^2 = L^2 \left[ -r^2 dt^2 + \frac{dr^2}{r^2} + r^2 d\theta^2 \right].$$

Then, masses and angular momenta of these black holes for the Killing vectors  $\xi_T = \frac{1}{L} \frac{\partial}{\partial t}$  and  $\xi_R = \pm \frac{\partial}{\partial \theta}$  are given by the so-called ADT charge  $Q^{\mu\nu}$  as<sup>3</sup>

$$\begin{aligned} M &= \frac{1}{4G} \sqrt{-\det g} \, Q_R^{rt}(\xi_T) \Big|_{r \rightarrow \infty} = \frac{1}{8G} \Delta_0, \\ J &= \frac{1}{4G} \sqrt{-\det g} \, Q_R^{rt}(\xi_R) \Big|_{r \rightarrow \infty} = \pm \frac{L}{8G} \Delta_0. \end{aligned} \quad (25)$$

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<sup>3</sup>See [22] for our notation and some details about ADT formalism. Note also that  $\mathcal{E}_{\mu\nu}$  expression only enters in the computation of the ADT charge and the so-called the correction term or hybrid formalism by Cl  ment [23] is not needed in this case since all our solutions satisfy the standard Brown-Henneaux boundary conditions.

Therefore, masses and the total angular momenta satisfy the extremal relation in these black holes as  $ML = \pm J$ . This relation strongly suggests that any black hole solution obtained from our reduced EOM is stable since the bound for angular momentum is saturated.

Now, we argue that the inequality  $W(\phi_H) \geq \mathcal{W}(\phi_\infty) = 2$  holds in general, which is an important ingredient to show the consistency with the holographic  $c$ -theorem. On general ground, it is natural to think that masses of hairy black holes deformed from BTZ black holes by a scalar field are always greater than those of hairless BTZ black holes since the scalar hair produces another contribution to masses. By accepting this assumption on mass inequality between extremal hairy black holes and extremal BTZ black holes, one can see that

$$M(\text{hairy}) = \frac{\Delta_0}{8G} = \frac{r_H^2}{8G} \mathcal{W}(\phi_H) \geq M(\text{BTZ}) = \frac{r_H^2}{4G},$$

which implies that

$$\mathcal{W}(\phi_H) \geq \mathcal{W}(\phi_\infty) = 2. \quad (26)$$

The Bekenstein-Hawking-Wald entropy of the above extremal black holes can be read from the Wald formula or the area law as

$$\mathcal{S}_{BHW} = \frac{A_H}{4G} = \frac{\pi \bar{L} \bar{r}_H}{2G} = \frac{\pi L r_H}{2G}, \quad (27)$$

and the Hawking temperature of these black holes are always zero because of the extremality. This nature of zero Hawking temperature also indicates the stability of the hairy extremal black holes. The angular velocity of these black holes at the horizon  $r_H$  is given by

$$\Omega_H = \frac{1}{L} \left[ C_{\mp} \mp \frac{1}{r_H} e^{A(r_H)} \right]. \quad (28)$$

Since the angular velocity of these black holes with  $C_{\mp} = \pm 1$  is given by  $\Omega_H = \pm(1/L)$  and the Hawking temperature  $T_H$  is zero, one can check that the first law of black hole thermodynamics is satisfied trivially.

### 2.3 Analytic solutions

In this section, we present analytic solutions of our first order reduced EOM. According to the given setup, one can try to solve the last differential equation (14) to obtain  $r = r(\phi)$  for the given superpotential  $\mathcal{W}$ . Then, one can determine metric functions  $A$ ,  $B$  and  $C$  just by Eqs. (11) and (15). In principle, this is the correct way to obtain solutions. However, it is not easy to obtain *analytic* solutions in this way. Therefore, we take a slightly different route: we will try to solve this equation by taking  $r$  as an appropriate function of  $\phi$ . To find exact solutions of the above first order reduced EOM, let us try the simplest choice for  $r^2$  just as the asymptotic form itself:

$$r^2 = \frac{4\omega_1}{(\phi - \phi_\infty)^2}. \quad (29)$$

Inserting this ansatz in the reduced differential equation (14), one can see that the superpotential  $\mathcal{W}$  is given by

$$\mathcal{W} = \alpha \left[ 4 + (\phi - \phi_\infty)^2 \right] + \beta e^{(\phi - \phi_\infty)^2/4}, \quad (30)$$

where the constant  $\alpha$  is related to  $\Delta_0$  in Eq. (14) as  $\Delta_0 = 4\alpha\omega_1$  and  $\beta$  is an arbitrary constant. The metric functions  $A$ ,  $B$  and  $C$  are determined in terms of  $\phi$  through Eq. (15), which can be converted to the functions of  $r$  as

$$e^A = r \left[ 2\alpha e^{-\omega_1/r^2} + \frac{\beta}{2} \right], \quad e^B = e^{-\omega_1/r^2} e^{-A}, \quad e^C = \mp \frac{1}{r} e^A, \quad (31)$$

where we have rescaled the time coordinate as usual to absorb the integration constant appropriately such that the asymptotic boundary condition on  $A$  is satisfied. Asymptotic boundary conditions on metric functions also lead to  $2\alpha + \beta/2 = 1$ . Now, let us impose the existence of horizon through  $e^{-B(r_H)} = 0$ , which leads to

$$2\alpha + \frac{\beta}{2} e^{(\phi(r_H) - \phi_\infty)^2/4} = 0.$$

Then, one can obtain  $\alpha$ ,  $\beta$  in terms of  $r_H^2 = 4\omega_1/(\phi_H - \phi_\infty)^2$  as

$$\alpha = \frac{1}{2} \frac{1}{1 - e^{-\omega_1/r_H^2}}, \quad \beta = -\frac{2e^{-\omega_1/r_H^2}}{1 - e^{-\omega_1/r_H^2}}. \quad (32)$$

At last, one can see that black hole solutions given by  $A(r)$ ,  $B(r)$ ,  $C(r)$  and  $\phi(r)$  are nothing but those in Ref. [13], which we have obtained in a different way using the first order reduced EOM.

One can identify the near horizon geometry of these black holes with the self-dual orbifold of  $AdS_3$  with the radius  $\bar{L}$  which is rescaled from the asymptotic radius  $L$  as

$$\bar{L} \equiv L \left[ \frac{1 - e^{-\omega_1/r_H^2}}{\omega_1/r_H^2} \right]. \quad (33)$$

To see this fact, one may use a new radial coordinate  $\rho$  defined by

$$\rho \equiv e^{-\omega_1/r_H^2} \left[ \frac{2\omega_1/r_H^2}{1 - e^{-\omega_1/r_H^2}} \right]^2 (r - r_H),$$

and then one can explicitly check that the metric takes the form given in Eq.(23). This shows us that the above hairy extremal AdS black holes interpolate between the asymptotic  $AdS_3$  with the radius  $L$  and the self-dual orbifold of  $AdS_3$  with the radius  $\bar{L}$ .

Now, let us take a look at another analytic solution of our first order equations. Under the successful reproduction of known solutions, one may try another choice for  $r^2$  as

$$r^2 = \frac{4\omega_1}{\sinh^2(\phi - \phi_\infty)}. \quad (34)$$

Following the same procedure in the above, one obtains

$$W(\phi) = \alpha \left[ 4 + \sinh^2(\phi - \phi_\infty) \right] + \beta \left[ \cosh(\phi - \phi_\infty) \right]^{1/2},$$

where we have used the Eq.(14) to obtain this result. The constant  $\alpha$  is related to  $\Delta_0 (= -12\alpha\omega_1)$ , as in the previous case. Then, metric functions  $A$ ,  $B$  and  $C$  are given by

$$e^A = r \left[ 2\alpha \left( 1 + \frac{4\omega_1}{r^2} \right)^{\frac{3}{4}} + \frac{\beta}{2} \right], \quad e^B = \left( 1 + \frac{4\omega_1}{r^2} \right)^{-\frac{1}{4}} e^{-A}, \quad e^C = \mp \frac{1}{r} e^A. \quad (35)$$

Asymptotic boundary conditions on  $A$  and  $B$  with the existence of the horizon lead to

$$\alpha = -\frac{1}{2} \frac{1}{\left( 1 + \frac{4\omega_1}{r_H^2} \right)^{\frac{3}{4}} - 1}, \quad \beta = \frac{2 \left( 1 + \frac{4\omega_1}{r_H^2} \right)^{\frac{3}{4}}}{\left( 1 + \frac{4\omega_1}{r_H^2} \right)^{\frac{3}{4}} - 1}. \quad (36)$$

These are new extremal AdS black hole solutions as far as the authors know. By introducing

$$\rho \equiv \left( 1 + \frac{4\omega_1}{r_H^2} \right)^{-\frac{1}{4}} \left[ \frac{6\omega_1/r_H^2}{\left( 1 + 4\omega_1/r_H^2 \right)^{3/4} - 1} \right]^2 (r - r_H),$$

one can show that the near horizon geometry is given by self-dual orbifold of  $AdS_3$  as the same form with Eq.(23) with  $\bar{L}$  and  $\bar{r}_H$  defined by

$$\bar{L} \equiv L \left[ \frac{\left( 1 + 4\omega_1/r_H^2 \right)^{3/4} - 1}{3\omega_1/r_H^2} \right], \quad \bar{r}_H \equiv r_H \left[ \frac{L}{\bar{L}} \right]. \quad (37)$$

As the final example, we take the following ansatz

$$r^2 = \frac{4\omega_1}{\sin^2(\phi - \phi_\infty)}. \quad (38)$$

Then, the superpotential and metric functions are given by

$$\begin{aligned} \mathcal{W} &= \alpha \left[ 4 + \sin^2(\phi - \phi_\infty) \right] + \beta \cos^{-\frac{1}{2}}(\phi - \phi_\infty), \\ e^A &= r \left[ 2\alpha \left( 1 - \frac{4\omega_1}{r^2} \right)^{5/4} + \frac{\beta}{2} \right], \quad e^B = \left( 1 - \frac{4\omega_1}{r^2} \right)^{1/4} e^{-A}, \quad e^C = \mp \frac{1}{r} e^A. \end{aligned} \quad (39)$$

and the constants  $\alpha$  and  $\beta$  are determined as

$$\alpha = \frac{1/2}{1 - \left( 1 - \frac{4\omega_1}{r_H^2} \right)^{5/4}}, \quad \beta = -\frac{2 \left( 1 - \frac{4\omega_1}{r_H^2} \right)^{5/4}}{1 - \left( 1 - \frac{4\omega_1}{r_H^2} \right)^{5/4}}. \quad (40)$$

Using the radial coordinate  $\rho$  in this case as

$$\rho \equiv \left( 1 - \frac{4\omega_1}{r_H^2} \right)^{\frac{1}{4}} \left[ \frac{10\omega_1/r_H^2}{1 - \left( 1 - 4\omega_1/r_H^2 \right)^{5/4}} \right]^2 (r - r_H),$$

one can check that the near horizon geometry is once again given by self-dual orbifold of  $AdS_3$  in Eq. (23) with the radius  $\bar{L}$  and  $\bar{r}_H$

$$\bar{L} \equiv L \left[ \frac{1 - (1 - 4\omega_1/r_H^2)^{5/4}}{5\omega_1/r_H^2} \right], \quad \bar{r}_H \equiv r_H \left[ \frac{L}{\bar{L}} \right]. \quad (41)$$

Some comments for the above black hole solutions are in order. Firstly, one may note that there is a new free parameter in the above solutions denoted as  $\omega_1$  which is related to the scalar field value at the horizon,  $\phi_H$ , and does not exist in extremal BTZ black holes. As is obvious from our method, this parameter  $\omega_1$  is also related to the coefficient of the leading term in the superpotential  $\mathcal{W}$ . Secondly, one may note that the above solutions are the extension of extremal BTZ black holes to hairy cases and reduce to extremal BTZ black hole solutions when the scalar field is turned off. To see this explicitly, one should take  $\omega_1 \rightarrow 0$  with the frozen scalar field  $\phi = \phi_\infty$ . Then, all the above expressions of black hole solutions reduce to those of extremal BTZ black holes. This reveals that the presence of a scalar field may produce diverse hairy black hole solutions via a scalar potential, which reduce to the same BTZ black holes when a scalar field is turned off. This point will also be important to understand the nature of black hole solutions in NMG, which are presented only in the perturbative form in the next section.

One may wonder about using the so-called Fefferman-Graham coordinates in this case. That is to say, a new radial coordinate  $\eta$  may be introduced by

$$d\eta = e^{B(r)} dr.$$

This gives us the so-called FG coordinates useful in later sections and corresponds to taking the following form of the metric ansatz

$$ds^2 = L^2 \left[ -e^{2A(\eta)} dt^2 + d\eta^2 + e^{2R(\eta)} \left( d\theta + e^{C(\eta)} dt \right)^2 \right]. \quad (42)$$

In these coordinates, most of the first order reduced EOM in the AdS-Schwarzschild coordinates remain as first order differential equations<sup>4</sup> (recall that  $r \equiv e^{R(\eta)}$ )

$$\dot{\phi} = -\partial_\phi \mathcal{W}, \quad \dot{A} + \dot{R} = \mathcal{W}, \quad \dot{C} = \dot{A} - \dot{R}, \quad (43)$$

where the dot denotes the differentiation with respect to the radial coordinate  $\eta$ . However, the first order differential equation for  $B(r)$  is transformed to the second order one for  $R(\eta)$  (or for  $\lambda \equiv e^{2R}$ ) as

$$\ddot{\lambda} - \dot{\phi}(\partial_\phi \mathcal{W})\lambda - \mathcal{W}\dot{\lambda} = 0.$$

Note that this second order differential equations is equivalent to  $E_{\theta\theta} = 0$  in these coordinates and corresponds to Eq. (13) in the  $(r, t, \theta)$  coordinates. This means that the reduced EOM may be taken by the first order equations for  $\phi$ ,  $A+R$  and  $C$  given in Eq. (43), together with  $E_{\theta\theta} = 0$ . Interestingly, the second differential equation for  $R$  (or  $E_{\theta\theta} = 0$ ) can be integrated into the first order form as

$$\lambda \mathcal{W} - \dot{\lambda} = \text{constant} = \Delta_0,$$

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<sup>4</sup>As in the AdS-Schwarzschild coordinates, we take the integration constant for  $C$  such as  $C = A - R$ .

which corresponds to the Eq. (14) in  $(r, t, \theta)$  coordinates. It is also interesting to note that this integrated first order equation is automatically satisfied for domain wall solutions which are included in our reduced EOM as the special case given by

$$C = 0, \quad \dot{A} = \dot{R} = \frac{1}{2}\mathcal{W}, \quad \Delta_0 = 0. \quad (44)$$

This is consistent with our interpretation of the constant  $\Delta_0$  as related to the near horizon of black holes, which should be absent in domain walls. Though these coordinates cover only outside the horizon and do not seem so useful in obtaining analytic solutions of AdS black holes in Einstein gravity, those simplify some computations and turn out to be particularly useful in NMG and in the holographic  $c$ -theorem, which are presented in next sections.

### 3 Extremal Black Hole Solutions in NMG

In this section we obtain lower order reduced EOM in NMG and then perturbative black hole solutions in NMG with an interacting scalar field, which reduce to extremal BTZ black holes when the scalar field is turned off.

#### 3.1 New massive gravity with a scalar field

New massive gravity(NMG) is a three-dimensional higher curvature gravity introduced as the covariant completion of Pauli-Fierz massive graviton theory [14]. Later it was recognized that NMG is more or less the unique extension of Einstein gravity consistent with the holographic  $c$ -theorem [7, 15]. In our convention, the Lagrangian of NMG with a scalar field is given by

$$S = \frac{1}{16\pi G} \int d^3x \sqrt{-g} \left[ \sigma R + \frac{1}{m^2} \mathcal{K} - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right], \quad (45)$$

where  $\sigma$  takes 1 or  $-1$ . The parameter  $m^2$  can take positive or negative values, and  $\mathcal{K}$  is a specific combination of scalar curvature square and Ricci tensor square defined by

$$\mathcal{K} = R_{\mu\nu} R^{\mu\nu} - \frac{3}{8} R^2. \quad (46)$$

The equations of motion(EOM) of NMG are given by

$$0 = E_{\mu\nu} \equiv \mathcal{E}_{\mu\nu} - T_{\mu\nu}, \quad (47)$$

where

$$\mathcal{E}_{\mu\nu} \equiv \sigma \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) + \frac{1}{2m^2} \mathcal{K}_{\mu\nu}, \quad T_{\mu\nu} \equiv \frac{1}{2} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \left[ \frac{1}{2} \partial_\alpha \phi \partial^\alpha \phi + V(\phi) \right],$$

and  $\mathcal{K}_{\mu\nu}$ , using  $\mathcal{D}_\mu$  as a covariant derivative with respect to  $g_{\mu\nu}$ , is defined by

$$\mathcal{K}_{\mu\nu} = g_{\mu\nu} \left( 3R_{\alpha\beta} R^{\alpha\beta} - \frac{13}{8} R^2 \right) + \frac{9}{2} R R_{\mu\nu} - 8R_{\mu\alpha} R^\alpha_\nu + \frac{1}{2} \left( 4\mathcal{D}^2 R_{\mu\nu} - \mathcal{D}_\mu \mathcal{D}_\nu R - g_{\mu\nu} \mathcal{D}^2 R \right).$$

The equation of motion for the scalar field  $\phi$  takes the same form with Einstein gravity given in Eq.(5). We will focus on  $\sigma = 1$  case, and we set  $\sigma = 1$  in the following.

### 3.2 Black hole solutions in new massive gravity

As was done in the domain walls in NMG [16], let us introduce the superpotential in NMG such that the scalar potential is given by

$$V(\phi) = \frac{1}{2L^2} \left( \frac{\partial \mathcal{W}}{\partial \phi} \right)^2 \left[ 1 - \frac{1}{8m^2 L^2} \mathcal{W}^2 \right]^2 - \frac{1}{2L^2} \mathcal{W}^2 \left[ 1 - \frac{1}{16m^2 L^2} \mathcal{W}^2 \right]. \quad (48)$$

This generalized form of potential in terms of superpotential was first considered by Low and Zee in the context of scalar field coupled to higher derivative gravity [11]. Motivated by results in Einstein gravity as given in Eq.(43), let us take the first order equations for  $A(\eta)$  and  $C(\eta)$

$$\dot{\phi} = -\frac{\partial \mathcal{W}}{\partial \phi} \left[ 1 - \frac{1}{8m^2 L^2} \mathcal{W}^2 \right], \quad \dot{A} + \dot{R} = \mathcal{W}, \quad \dot{C} = \dot{A} - \dot{R}. \quad (49)$$

As in Einstein gravity, the last equation for  $C$  can be trivially integrated and may be omitted in the following.

One can check that these equations solve scalar EOM,  $E_\phi = 0$  and metric EOM,  $E_{\mu\nu} = 0$  except  $E_{\theta\theta} = 0$ , even in NMG. Explicitly in these coordinates, the scalar EOM and metric EOM except  $E_{\theta\theta} = 0$  can be shown to be satisfied as follows:

$$\begin{aligned} E_\phi &= \ddot{\phi} + \mathcal{W} \dot{\phi} - L^2 \partial_\phi V = 0, \\ -E_{\eta\eta} &= L^2 V + \frac{1}{2} \mathcal{W}^2 \left[ 1 - \frac{1}{16m^2 L^2} \mathcal{W}^2 \right] + \dot{\phi} \partial_\phi \mathcal{W} \left[ 1 - \frac{1}{8m^2 L^2} \mathcal{W}^2 \right] + \frac{1}{2} \dot{\phi}^2 = 0, \\ -e^{-A-R} E_{t\theta} &= L^2 V + \frac{1}{2} \mathcal{W}^2 \left[ 1 - \frac{1}{16m^2 L^2} \mathcal{W}^2 \right] + \frac{1}{2} \dot{\phi} \partial_\phi \mathcal{W} \left[ 1 - \frac{1}{8m^2 L^2} \mathcal{W}^2 \right] = 0. \end{aligned} \quad (50)$$

Using  $\lambda \equiv e^{2R}$ , one can represent  $E_{\theta\theta}$  as

$$\begin{aligned} -2E_{\theta\theta} &= \frac{1}{m^2 L^2} \left( \ddot{\lambda} - 2\dot{\lambda} \dot{\mathcal{W}} \right) + \dot{\lambda} \left[ 1 + \frac{9}{2m^2 L^2} \left( \frac{1}{4} \mathcal{W}^2 - \dot{\phi} \partial_\phi \mathcal{W} \right) \right] \\ &\quad + \dot{\lambda} H_1(\mathcal{W}, \partial_\phi \mathcal{W}, \dots) + \lambda H_2(\mathcal{W}, \partial_\phi \mathcal{W}, \dots), \end{aligned} \quad (51)$$

where  $H_1$  and  $H_2$  are some functions of  $\mathcal{W}$  and its derivative with respect to  $\phi$  given by

$$\begin{aligned} H_1 &= -\frac{\mathcal{W}}{2} \left( 1 + \frac{\mathcal{W}^2}{8m^2 L^2} \right) - \frac{1}{8m^2 L^2} \left[ 14 \left( \dot{\phi}^2 \partial_\phi^2 \mathcal{W} + \ddot{\phi} \partial_\phi \mathcal{W} \right) - 15 \mathcal{W} \dot{\phi} \partial_\phi \mathcal{W} \right] \\ H_2 &= -L^2 V - \frac{\mathcal{W}^2}{2} \left( 1 - \frac{\mathcal{W}^2}{16m^2 L^2} \right) - \frac{\dot{\phi}^2}{4m^2 L^2} \left( 3(\partial_\phi \mathcal{W})^2 + 3\mathcal{W} \partial_\phi^2 \mathcal{W} - 2\dot{\phi} \partial_\phi^3 \mathcal{W} \right) \\ &\quad + \frac{1}{4m^2 L^2} \left[ \left( \mathcal{W}^2 \partial_\phi \mathcal{W} + 6 \partial_\phi^2 \mathcal{W} \ddot{\phi} \right) \dot{\phi} + \partial_\phi \mathcal{W} \left( 3\mathcal{W} \ddot{\phi} - 2\dot{\phi} \ddot{\phi} \right) \right]. \end{aligned}$$

Now, one can see that the equation  $E_{\theta\theta} = 0$  can be integrated as

$$\tilde{\Delta}_0 = \frac{1}{m^2 L^2} (\ddot{\Psi} - \mathcal{W} \dot{\Psi}) + \left[ 1 + \frac{1}{8m^2 L^2} (\mathcal{W}^2 - 4\dot{\mathcal{W}}) \right] \Psi, \quad (52)$$

where  $\tilde{\Delta}_0$  denotes the integration constant and  $\Psi$  is defined as

$$\Psi \equiv \lambda \mathcal{W} - \dot{\lambda}.$$

The physical meaning of the constant  $\tilde{\Delta}_0$  turns out to be similar to that of  $\Delta_0$  in Einstein gravity. That is to say, it is related to the conserved charges of black holes and their near horizon geometry, which will be shown in the below. Note also that we have reduced the fourth order EOM effectively to the second order one.

By transforming  $(\eta, t, \theta)$  coordinates to  $(\phi, t, \theta)$  coordinates, (which corresponds to taking  $\phi$  as the radial coordinate instead of  $\eta$ ) as in Einstein gravity, which corresponds to the following change of variables

$$\frac{\partial}{\partial \eta} = \dot{\phi} \partial_{\phi} = -(\partial_{\phi} \mathcal{W}) \left[ 1 - \frac{1}{8m^2 L^2} \mathcal{W}^2 \right] \partial_{\phi}, \quad (53)$$

where we have used the reduced EOM for  $\phi$  in the second equality. Through this transformation,  $\Psi$  is represented as

$$\Psi = r^2 \mathcal{W} + (\partial_{\phi} r^2) (\partial_{\phi} \mathcal{W}) \left[ 1 - \frac{1}{8m^2 L^2} \mathcal{W}^2 \right], \quad (54)$$

which should satisfy the differential equation (52). When  $\Psi$  is obtained, metric functions  $A$  and  $B$  can be given by

$$\begin{aligned} A &= -\ln r + \int^{\phi} d\phi' e^B \mathcal{W} \partial_{\phi'} r, \\ e^{-B} &= -(\partial_{\phi} r) (\partial_{\phi} \mathcal{W}) \left[ 1 - \frac{1}{8m^2 L^2} \mathcal{W}^2 \right] = \frac{r}{2} \left[ \mathcal{W} - \frac{1}{r^2} \Psi \right]. \end{aligned} \quad (55)$$

These expressions for metric functions in  $(r, t, \theta)$  coordinates come from the first order reduced EOM for  $\phi$  and  $A + R$  in (49), through the substitution of  $\partial/\partial \eta$  and  $R$  by  $e^{-B} \partial/\partial r$  and  $\ln r$ .

The differential equation for  $\Psi$  is a nonlinear inhomogeneous equation. It is not easy to obtain its solution analytically except a trivial case. Therefore, we try to obtain asymptotic series form of solutions in NMG case, which might be sufficiently illuminating for discussion of holographic c-theorem in this case. As alluded in the previous section, the scalar hairy black holes with asymptotic AdS space and with the near horizon AdS space would correspond to the class of black holes which reduce to extremal BTZ black holes even in NMG. Before doing these perturbative analysis, let us consider the cases which allow analytic results.

Firstly, as in the Einstein gravity case, domain wall solutions correspond to

$$C = 0, \quad \dot{A} + \dot{R} = \mathcal{W}, \quad \Psi = 0, \quad \tilde{\Delta}_0 = 0,$$

which allow analytic results and were studied in Ref. [16].



Secondly, as a trivial example, let us check that extremal BTZ black holes are solutions of the above differential equation of  $\Psi$ . By taking  $\mathcal{W} = 2$ , the first order reduced EOM for the scalar field  $\phi$ , and metric function  $\dot{A} + \dot{R}$ , lead to

$$V = -\frac{2}{\ell^2}, \quad \dot{\phi} = 0, \quad \dot{A} + \dot{R} = 2, \quad (56)$$

where  $\ell$  is defined by

$$\frac{1}{\ell^2} \equiv \frac{1}{L^2} \left[ 1 - \frac{1}{4m^2 L^2} \right].$$

The second order differential equation for  $\Psi$  gives us a constant  $\Psi$  as

$$\Psi = 2\lambda - \dot{\lambda} = 2r_H^2, \quad (57)$$

where we have introduced the constant  $r_H$  as  $2r_H^2 \equiv \tilde{\Delta}_0[1 + 1/2m^2 L^2]^{-1}$ . This gives us  $e^{2R} = e^{2\eta} + r_H^2$ , which corresponds to the well-known extremal BTZ black hole solutions in NMG with the horizon radius  $r_H$ .

To see nontrivial solutions one may try to solve the above equations for a given superpotential. However, one can see that the resulting solution for  $\Psi$  is given by a complicated function. Furthermore, it is not easy to obtain analytic form of metric and scalar field in this way. As in Einstein gravity, we perform the perturbative calculation at the asymptotic infinity and on the near horizon. This analysis already reveals important features of black hole solutions and is sufficient to verify that those black holes are extremal ones.

Since the methodology is completely identical with Einstein gravity case, we briefly present the intermediate steps. In summary, let us consider the following asymptotic expansions for metric variables, the superpotential and the scalar field:

$$\begin{aligned} A(r) &= \ln r + \tilde{a}_1 r^{-2} + \dots, & B(r) &= -\ln r + \tilde{b}_1 r^{-2} + \dots, \\ \mathcal{W} &= 2 + \frac{2\tilde{\omega}_1}{r^2} + \dots = 2 + \frac{1}{2q}(\phi - \phi_\infty)^2 + \dots, \\ \phi(r) &= \phi_\infty + \frac{\tilde{\phi}_1}{r} + \dots, \end{aligned} \quad (58)$$

where  $q$  denotes

$$q \equiv 1 - \frac{1}{2m^2 L^2}.$$

It turns out that  $\tilde{a}_1$  and  $\tilde{b}_1$  satisfy  $\tilde{a}_1 + \tilde{b}_1 = -\tilde{\omega}_1$  and  $\tilde{\phi}_1^2 = 4q\tilde{\omega}_1$ . By using the expansion of  $\Psi$  in terms of  $r$  as

$$\Psi = \Psi_0 + \frac{\Psi_1}{r^2} + \dots, \quad (59)$$

and by solving the Eq. (52) perturbatively in terms of  $r$ , one obtains the following

$$\Psi_0 \left[ 1 + \frac{1}{2m^2 L^2} \right] = \tilde{\Delta}_0.$$

One can also obtain through Eq. (55)

$$\tilde{a}_1 = -\frac{1}{2}\tilde{\Delta}_0\left[1 + \frac{1}{2m^2L^2}\right]^{-1}, \quad \tilde{b}_1 = \frac{1}{2}\tilde{\Delta}_0\left[1 + \frac{1}{2m^2L^2}\right]^{-1} - \tilde{\omega}_1. \quad (60)$$

Note that this takes the form of hairy deformation from BTZ black holes given in the Eq. (57). Masses and angular momenta of these black holes can be obtained by the ADT method. Using the results given in Ref. [22], one can see that

$$M = \frac{1}{8G}\tilde{\Delta}_0, \quad J = \pm \frac{L}{8G}\tilde{\Delta}_0, \quad (61)$$

which satisfy the extremal condition  $ML = \pm J$ , as in Einstein gravity (see Ref. [22] for more details about ADT charges in NMG and how these give the above results.). As in Einstein gravity, it is straightforward to argue in NMG that the inequality,  $\mathcal{W}(\phi_H) \geq \mathcal{W}(\phi_\infty)$  holds in general from mass inequality between hairy deformed extremal BTZ black holes and hairless ones,  $M(\text{hair}) \geq M(\text{BTZ})$ .

Now, let us consider the expansions on the near horizon. By doing the perturbative analysis on the near horizon, one can see that

$$\tilde{\Delta}_0 = r_H^2 \mathcal{W}(\phi_H) \left[1 + \frac{1}{8m^2L^2} \mathcal{W}^2(\phi_H)\right]. \quad (62)$$

By expanding the radial coordinate  $r$  and the superpotential  $\mathcal{W}$  in terms of  $\phi$  as

$$\begin{aligned} r &= r_H + \tilde{h}_0(\phi - \phi_H) + \dots, \\ \mathcal{W}(\phi) &= \mathcal{W}(\phi_H) - \frac{1}{2}\mathcal{W}(\phi_H) \left[1 - \frac{1}{8m^2L^2} \mathcal{W}^2(\phi_H)\right]^{-1} (\phi - \phi_H) + \dots, \end{aligned}$$

which is important to see the relation between  $\tilde{\Delta}_0$  and  $\mathcal{W}(\phi_H)$ , one can also obtain metric functions as, through the perturbative analysis,

$$e^{A(r)} = \tilde{s}_0 \mathcal{W}(\phi_H)(r - r_H) + \dots, \quad e^{B(r)} = \frac{1}{\mathcal{W}(\phi_H)(r - r_H)} + \dots, \quad (63)$$

where  $\tilde{s}_0$  is a certain non-vanishing constant related to the specific black holes or the interval of integral. As in Einstein gravity, one can show that

$$\bar{L} = \frac{2L}{\mathcal{W}(\phi_H)}, \quad (64)$$

and can see that the extremality condition (21) is fulfilled. This result shows us that the black holes under the consideration are extremal ones, indeed. We also obtain the same results through the same expansions with original EOM without using the superpotential  $\mathcal{W}$  (see appendix B.)

Note that the Bekenstein-Hawking-Wald entropy of these extremal black holes are given by

$$\mathcal{S}_{BHW} = \frac{A_H}{4G} \left[1 + \frac{1}{2m^2L^2}\right], \quad A_H \equiv 2\pi L r_H. \quad (65)$$

One can also verify that these are consistent with the first law of black hole thermodynamics trivially as in Einstein gravity.

Let us consider the domain walls in this perturbative approach. By taking the asymptotic form of the superpotential  $\mathcal{W}$  in the same form with that in Einstein gravity as

$$\mathcal{W} = 2 + \frac{1}{2}(\phi - \phi_\infty)^2 + \dots,$$

one can obtain asymptotic expansion of various variables in NMG as

$$\begin{aligned} A(r) &= \ln r + \tilde{a}_1 r^{-2q} + \dots, & B(r) &= -\ln r + \tilde{b}_1 r^{-2q} + \dots, \\ \mathcal{W} &= 2 + \frac{2\tilde{\omega}_1}{r^{2q}} + \dots, & \phi(r) &= \phi_\infty + \frac{\tilde{\phi}_1}{r^q} + \dots, \end{aligned} \quad (66)$$

where  $\tilde{a}_1$  and  $\tilde{b}_1$  satisfy  $q\tilde{a}_1 + \tilde{b}_1 = -\omega_1$  and  $\tilde{\phi}_1^2 = 4\tilde{\omega}_1$ . For a generic expansion of  $\Psi$  in terms of  $\phi$ , one can see that all the coefficients of  $\Psi$  vanish by solving the Eq. (52) perturbatively, *i.e.*

$$\Psi = \tilde{\Delta}_0 = 0, \quad (67)$$

which corresponds to the domain wall case. This short computation shows us that the asymptotic form of the superpotential for domain wall solutions in NMG should be taken differently from those of black holes and partially explains to us why the reduced EOM for domain walls are different from those for black holes.

## 4 Holographic C-Theorem

In this section we consider the holographic  $c$ -theorem in the context of extremal AdS black holes in three dimensional Einstein gravity and in NMG. By constructing central charge flow functions holographically, one finds some non-trivial checks of the consistency between central charge expressions and parameters in extremal AdS black hole solutions.

A central charge flow function of the boundary field theory dual to Einstein gravity may be introduced as

$$C(\phi) = \frac{3L}{G} \frac{1}{\mathcal{W}(\phi)}, \quad (68)$$

which gives us the central charge values on the asymptotic AdS space and on the near horizon geometry as

$$\begin{aligned} C(\phi \rightarrow \phi_\infty) &= c_{UV} = \frac{3L}{G} \frac{1}{\mathcal{W}(\phi_\infty)} = \frac{3L}{2G}, \\ C(\phi \rightarrow \phi_H) &= c_{IR} = \frac{3L}{G} \frac{1}{\mathcal{W}(\phi_H)} \equiv \frac{3L_{IR}}{2G}, \end{aligned} \quad (69)$$

where we have introduced a IR scale  $L_{IR} \equiv 2L/\mathcal{W}(\phi_H)$ . Note that the superpotential value at the horizon,  $\mathcal{W}(\phi_H)$ , is always greater than its asymptotic value  $\mathcal{W}(\phi_\infty) = 2$ , which implies that  $c_{UV} \geq c_{IR}$ . Moreover,  $c_{UV}$  takes the standard value of two-dimensional boundary field theory dual to  $AdS_3$  with the radius  $L$ , which can be obtained by Brown-Henneaux method [18] or the standard AdS/CFT dictionary [24, 25, 26]. Furthermore, in the domain wall limit, Eq. (44), the central charge flow function reduces to the well-known form  $C(\eta) = 3L/2G \cdot 1/A(\eta)$ .

In the standard dictionary of the AdS/CFT correspondence, one identifies the IR scale  $L_{IR}$  with the near horizon scale  $\bar{L}$  from the geometry. To verify the consistency of our choice of central charge flow functions, one needs to check that the central charge flow functions reproduce the central charges of dual conformal field theories even at the IR conformal point. Since  $\bar{L}$  is already determined by black hole parameters and  $L_{IR}$  is done by the superpotential values at the horizon, two results should be matched in order for the self-consistency of our construction. As was shown in Eq. (24), the expressions of  $\bar{L}$  is indeed identical with  $L_{IR}$  as

$$\bar{L} = L_{IR} = \frac{2L}{\mathcal{W}(\phi_H)} = \frac{2r_H^2 L}{\Delta_0}. \quad (70)$$

To see this explicitly for analytic solutions, one may note that for each ansatz of the radial coordinate  $r$  in terms of  $\phi$

$$r^2 = \frac{4\omega_1}{(\phi - \phi_\infty)^2}, \quad \frac{4\omega_1}{\sinh^2(\phi - \phi_\infty)}, \quad \frac{4\omega_1}{\sin^2(\phi - \phi_\infty)},$$

the superpotential values at the horizon,  $\mathcal{W}(\phi_H)$ , are given respectively by

$$\mathcal{W}(\phi_H) = \frac{2\omega_1/r_H^2}{1 - e^{-\omega_1/r_H^2}}, \quad \frac{6\omega_1/r_H^2}{(1 + \frac{4\omega_1}{r_H^2})^{3/4} - 1}, \quad \frac{10\omega_1/r_H^2}{1 - (1 - \frac{4\omega_1}{r_H^2})^{5/4}},$$

which are consistent with the general analysis as can be seen from Eqs.(33), (37) and (41).

Since we have verified that our central charge flow functions lead to the correct central charges at conformal end points, we turn to show their monotonic properties. According to the AdS/CFT correspondence, it is well known that the scale in the RG flow of dual field theory corresponds to the radial coordinate in the so-called Fefferman-Graham coordinates in the gravity side. To show the monotonic property of the above central charge function along the RG flow in the dual field theory, one needs to consider the derivative of the above central charge function with respect to the radial coordinate in the FG coordinates. The radial coordinate  $\eta$  introduced in Eq. (42) forms the FG coordinates together with  $(\theta, t)$  in our case. Note that  $\eta \rightarrow \infty$  corresponds to the asymptotic infinity (or UV) and  $\eta \rightarrow 0$  does to the near horizon (or IR). Now, one can see that

$$\frac{d}{d\eta} C(\phi(\eta)) = -\frac{3L}{G} \frac{1}{\mathcal{W}^2} (\partial_\phi \mathcal{W}) \dot{\phi} = \frac{3L}{G} \frac{1}{\mathcal{W}^2} (\partial_\phi \mathcal{W})^2 \geq 0, \quad (71)$$

where we have used the first order equation for  $\dot{\phi}$  which is given in these coordinate as  $\dot{\phi} = -\partial_\phi \mathcal{W}$ . This result means that the central charge is always increased when  $\eta$  becomes increased (or when

the energy scale is increased), and this can be regarded as the holographic construction of two dimensional  $c$ -theorem beyond the domain wall geometry.

Central charge flow function in NMG may be defined as

$$C(\phi) = \frac{3L}{G} \frac{1}{\mathcal{W}(\phi)} \left[ 1 + \frac{1}{8m^2 L^2} \mathcal{W}(\phi)^2 \right], \quad (72)$$

which gives us central charge values at UV and IR as

$$\begin{aligned} C(\phi_\infty) &= c_{UV} = \frac{3L}{2G} \left[ 1 + \frac{1}{2m^2 L^2} \right], \\ C(\phi_H) &= c_{IR} = \frac{3L}{G} \frac{1}{\mathcal{W}(\phi_H)} \left[ 1 + \frac{1}{8m^2 L^2} \mathcal{W}^2(\phi_H) \right] \equiv \frac{3L_{IR}}{2G} \left[ 1 + \frac{1}{2m^2 L_{IR}^2} \right]. \end{aligned} \quad (73)$$

where we have introduced  $L_{IR} = 2L/\mathcal{W}(\phi_H)$  as in the case of Einstein gravity. One may note that, on the contrary to Einstein gravity,  $c_{UV}$  and  $c_{IR}$  are not proportional to the cosmological constants at the asymptotic infinity and at the near horizon. It is quite useful to check  $\bar{L} = L_{IR}$  for the whole consistency of our results. Indeed, one can see that this is the case by the asymptotic analysis given in the previous section as was shown in Eq. (64).

The monotonic property of the chosen central charge flow functions is anticipated in NMG. Using the first order equation for scalar field given in Eq. (49), one can verify this anticipation as

$$\dot{C} = -\frac{3L}{G} \frac{1}{\mathcal{W}^2} (\partial_\phi \mathcal{W}) \left[ 1 - \frac{1}{8m^2 L^2} \mathcal{W}^2 \right] \dot{\phi} = \frac{3L}{G} \frac{1}{\mathcal{W}^2} (\partial_\phi \mathcal{W})^2 \left[ 1 - \frac{1}{8m^2 L^2} \mathcal{W}^2 \right]^2 \geq 0, \quad (74)$$

which represents the holographic construction of  $c$ -theorem dual to extremal black holes in NMG.

Now, we give some comments about the relation between holographic  $c$ -theorem and null energy condition on matters. For domain walls, one usually choose a central charge flow function as a function of metric variables and then relate directly the derivative of central charge flow functions, through EOM, to the null energy condition on matters. On the contrary, the connection between the holographic  $c$ -theorem and null energy condition is more or less indirect in our case, because the chosen central charge function depends on superpotential not metric variables. By taking null vectors on our geometry in FG coordinates as  $\mathcal{N}_\pm = (N_\pm^t, N_\pm^\eta, N_\pm^\theta) = (1/L)(\pm 1, e^{-R}, -e^{-A})$ , one may check that null energy condition on a scalar field is satisfied as

$$T_{\mu\nu} \mathcal{N}_\pm^\mu \mathcal{N}_\pm^\nu = \frac{1}{L^2} \dot{\phi}^2 \geq 0.$$

Only after using reduced EOM and not the original EOM, one may relate  $\dot{C}$  to  $T_{\mu\nu} \mathcal{N}_\pm^\mu \mathcal{N}_\pm^\nu$ . But this is somewhat indirect connection to the null energy condition compared to the case of domain walls, which use only the original EOM. This indicates that explicit solutions of EOM may be needed to verify the holographic  $c$ -theorem for more complicated geometry with two asymptotic AdS spaces as the reduced first order EOM are necessary to show those theorems in our black hole cases.

In connection with the holographic  $c$ -theorem, it is interesting to consider the entropy of boundary dual CFT by Cardy formula. Since the bulk contains two  $AdS_3$  spaces at the asymptotic infinity and at the near horizon, there are two entropy functions,  $S_{UV}$  and  $S_{IR}$ . One of the natural questions about these entropies is that there is any relation to the Bekenstein-Hawking-Wald entropy of extremal black holes. The well-known relations between conserved charges in the bulk gravity and energies in the dual CFT at the asymptotic boundary are given by [27]

$$M = E_L + E_R, \quad J = L(E_L - E_R), \quad (75)$$

where  $E_L/E_R$  are left/right energy in the dual CFT and related to the so-called left/right temperatures as  $E_{L/R} = (\pi^2 L/6) T_{L/R}^2$ . The Cardy formula gives us the entropy  $S_{UV}$  in terms of these left/right temperatures as

$$S_{UV} = \frac{\pi^2 L}{3} (c_L T_L + c_R T_R). \quad (76)$$

Note that  $c = c_L = c_R$ , since we are dealing with the parity even theories. The extremal condition in the black hole side means that one of the left and right energy should vanish in the dual CFT, which can be deduced from the conserved charge relations. For definiteness, let us consider  $ML = J$  case. After some computation, one can see that the entropy of dual CFT at the asymptotic boundary is not less than that the Bekenstein-Hawking-Wald entropy of extremal black holes as<sup>5</sup>

$$\begin{aligned} S_{UV} &= \frac{\pi^2 L}{3} c_{UV} T_L = \pi \sqrt{\frac{2}{3} c_{UV} J} \\ &= \begin{cases} \frac{A_H}{4G} \left[ \frac{\mathcal{W}(\phi_H)}{\mathcal{W}(\phi_\infty)} \right]^{1/2} & \text{Einstein} \\ \frac{A_H}{4G} \left[ 1 + \frac{1}{2m^2 L^2} \right] \left[ \frac{\mathcal{W}(\phi_H)}{\mathcal{W}(\phi_\infty)} \frac{1 + \frac{1}{8m^2 L^2} \mathcal{W}^2(\phi_H)}{1 + \frac{1}{8m^2 L^2} \mathcal{W}^2(\phi_\infty)} \right]^{1/2} & \text{NMG} \end{cases} \\ &\geq S_{BHW}. \end{aligned} \quad (77)$$

Note that the inequality between  $S_{UV}$  and  $S_{BHW}$  is the direct consequence of the inequality between the superpotential values  $\mathcal{W}(\phi_H) \geq \mathcal{W}(\phi_\infty) = 2$ , which can be verified explicitly in analytic solutions and was argued to be the case in general through mass inequalities between mass of hairy deformed extremal BTZ black holes and hairless ones.

In Einstein gravity, mass and angular momentum at the horizon,  $\bar{M}$  and  $\bar{J}$  as quasi-local quantities were computed and shown to satisfy  $\bar{M}\bar{L} = \bar{J} = J$  [13] for specific extremal black holes. This computation for quasi-local quantities on the near horizon requires only the near horizon fall-off behavior of various variables, our extremal black holes would satisfy the same relations. One of the very interesting results in this quasi-local computation is that the angular momentum is invariant from the asymptotic infinity to the near horizon, which was also observed in thermodynamic approach to black holes in a finite region [29]. This invariance of angular momentum in Einstein gravity leads to the entropy of dual CFT on the near horizon through Cardy formula as

$$S_{IR} = \pi \sqrt{\frac{2}{3} c_{IR} J} = \frac{A_H}{4G} = S_{BHW}. \quad (78)$$

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<sup>5</sup>This fact is already observed in Refs. [13, 28] for particular black holes in Einstein gravity.

One may interpret this invariance of angular momentum along the bulk radial direction in the dual CFT side as follows. Along the RG flow in dual field theory, the running of central charge is the consequence of the change of the AdS radius, while the bulk gravitational constant,  $G$ , or the coupling  $m^2$  of higher curvature terms are invariant. From the bulk perspective in our three-dimensional case, the AdS radius is the unique candidate for the running variable, which is related to the scalar field. Based on dimensional reasoning, one can see that the dual CFT energy  $E_{L/R}$ , which are computed from gravity, should scale inversely to the AdS radius without anomalous scaling. Therefore, the angular momentum is invariant with scaling of the AdS radius.

By assuming the validity of the angular momentum invariance, one can compute the entropy of dual CFT on the near horizon in NMG as

$$\begin{aligned} S_{IR} &= \pi \sqrt{\frac{2}{3} c_{IR} J} = \frac{A_H}{4G} \left[ 1 + \frac{1}{8m^2 L^2} \mathcal{W}^2(\phi_H) \right] \\ &\geq \mathcal{S}_{BHW} = \frac{A_H}{4G} \left[ 1 + \frac{1}{8m^2 L^2} \mathcal{W}^2(\phi_\infty) \right]. \end{aligned}$$

It is very interesting to observe that the entropy on the near horizon is also greater than the Bekenstein-Hawking-Wald entropy of extremal black holes in NMG whenever the scalar hair is nontrivial<sup>6</sup>. In summary for all cases, one can see that the following inequalities always hold

$$S_{UV} \geq S_{IR} \geq \mathcal{S}_{BHW}.$$

## 5 Conclusion

We have discovered Bogomol'nyi type of lower order differential equations for Einstein gravity and for NMG, which we called reduced EOM, in the presence of a minimally coupled scalar field. More explicitly, the first order reduced EOM in Einstein gravity is given by Eq. (7) and the reduced EOM in NMG by Eqs. (49) and (52). Using these equations, we have obtained various analytic hairy black hole solutions which include the previously known example as a special case. We also showed that all these solutions are consistent with the holographic  $c$ -theorem.

The asymptotic space of all our solutions is AdS space with the radius  $L$  and the near horizon geometry is the so-called self-dual orbifold of  $AdS_3$  with the radius  $\bar{L}$ . After showing that the simplest case of our black hole solutions corresponds to the well-known extremal BTZ black holes, the extremality of our black hole solutions are shown explicitly. We also showed that our reduced EOM implies generically the extremality of any solution of these equations under some mild assumptions, the validity of the power series expansion. In Einstein gravity we have presented several analytic solutions with some generic perturbative treatment and in NMG we

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<sup>6</sup> We would like to make some cautionary remarks for this statement. Since the original Cardy formula requires ‘effective central charge’  $c_{eff} = c - 24h_0$  ( $h_0$ : lowest conformal weight), as was reviewed in [30], there is a possibility of  $S_{IR} = \mathcal{S}_{BHW}$  for nontrivial hair even in NMG just as in Einstein gravity.

have done some perturbative solutions. We have also identified various physical quantities of extremal black hole solutions and shown that conserved charges of these black holes are related directly to the integration constant  $\Delta_0$  and  $\tilde{\Delta}_0$  of the reduced EOM.

Motivated from the domain wall case, we have proposed the holographic central charge flow functions and verified that they coincide with central charges at the conformal end points and that they also satisfy the anticipated monotonic properties. We have also performed the consistency checks on the central charge flow functions by showing that the near horizon scale  $\bar{L}$ , which is read from the geometry, can be identified with the IR scale  $L_{IR}$  in the central charge flow functions.

There are some future directions to pursue further. First of all, our reduced EOM are just rewritten down in analogy with the domain wall case. It would be very interesting to derive our reduced EOM by complete squaring of a certain energy functional or by using fake supersymmetry. It is notable that the second order original EOM in Einstein gravity reduce to the first order ones while all of the reduced EOM in NMG are not first order ones. This is clearly contrasted to the domain wall case which is also described by the first order ones even in NMG. It would be very interesting to reproduce conserved charges for our extremal black holes consistently in Einstein gravity and NMG by other methods. Another interesting direction is to verify the stability of our extremal black holes directly. Though we showed that all the black hole solutions from our reduced EOM are extremal and argued that they are all stable by the generic thermodynamic consideration, we did not perform any dynamical stability analysis through some perturbations.

It is also interesting to prove the inequality  $\mathcal{W}(\phi_H) \geq \mathcal{W}(\phi_\infty)$  rigorously for any sensible black hole solutions, which is argued to be the case on physical ground. It would be also interesting to verify or disprove our conjecture about the invariance of angular momentum along RG flow in general. In Einstein gravity, the invariance is rather established in various cases. However, it is not checked explicitly in other gravity theories like NMG. It is also valuable to find more solutions or to study the complete integrability of our first order reduced EOM in Einstein gravity. Another intriguing direction is the study on couplings with more than one scalar field. For instance, it might be possible to obtain vortex black hole solutions when a complex scalar field is coupled. Finally, it would be very interesting to extend the correspondence between DLCQ of two-dimensional field theory and extremal black holes to our hairy cases.

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## Appendix

### A Equations of motion in NMG for some coordinate systems

In this section, we present EOM in NMG only since EOM in Einstein gravity can be obtained by taking  $m^2 L^2 \rightarrow \infty$  in the following expressions of EOM in NMG.

For our metric ansatz (3) with the relation (11), we present the metric EOM in the following since the scalar EOM is already given in Eq. (5).

$$\begin{aligned}
0 &= E_{rr} = -L^2 e^{2B} V - \frac{1}{2r} \left[ r A'^2 + (2 - 2r B') A' + r \phi'^2 - 2B' + 2r A'' \right] - \frac{1}{2r^2} \\
&\quad + \frac{e^{-2B}}{32m^2 L^2 r^4} (r A' + 1)^2 \left[ r (A' (r A' - 4r B' + 2) + 4r A'' - 4B') - 3 \right], \\
0 &= E_{\theta\theta} = -L^2 r^2 V - \frac{1}{2} e^{-2B} (r^2 A'^2 - 2r B' + 1) \\
&\quad + \frac{e^{-4B}}{32m^2 L^2 r^2} \left[ A'^4 r^4 - 24A''^2 r^4 + 96B'^3 r^3 - 8A'^3 (r B' - 1) r^3 + 80A'' r^2 \right. \\
&\quad + 8B'^2 (22r^2 A'' - 13) r^2 + 40B'' r^2 - 7 - 64A'' B'' r^4 + 88A''' r^3 \\
&\quad + 16A'''' r^4 + 16B''' r^3 + 2A'^2 r^2 (4A'' r^2 + 12B'' r^2 + 66B' r - 36B'^2 r^2 - 13) \\
&\quad - 4B' r (24A''' r^3 + 90A'' r^2 + 28B'' r^2 + 5) \\
&\quad - 8A' r^2 (12r^2 B'^3 - 44r B'^2 + (21 - 15A'' r^2 - 14B'' r^2) B' \\
&\quad \left. + r (13A'' + 16B'' + 3r A''' + 2r B''')) \right], \\
0 &= E_{t\theta} = \mp \frac{1}{2} e^A \left[ e^{-2B} (B' - A' (r A' - r B' + 2) - r A'') - 2L^2 r V \right] \\
&\quad \mp \frac{e^{A-4B}}{32m^2 L^2 r^3} (r A' + 1)^2 \left[ r (A' (r A' - 2r B' + 2) - 2B' + 2r A'') - 1 \right],
\end{aligned} \tag{A.1}$$

where  $'$  denotes differentiation with respect to the radial coordinate  $r$ .

For the metric ansatz (42) with the relation  $C = A - R$ , which is written in the so-called FG coordinates, the scalar field equation and EOM in NMG are given by

$$\begin{aligned}
0 &= E_\phi = \frac{1}{L^2} \left[ (\dot{A} + \dot{R}) \dot{\phi} + \ddot{\phi} \right] - \partial_\phi V, \\
0 &= E_{\eta\eta} = -V L^2 - \frac{1}{2} \left[ (\dot{A} + \dot{R})^2 + \dot{\phi}^2 + 2 (\ddot{A} + \ddot{R}) \right] \\
&\quad + \frac{1}{32m^2 L^2} (\dot{A} + \dot{R})^2 \left[ (\dot{A} + \dot{R})^2 + 4 (\ddot{A} + \ddot{R}) \right],
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
0 &= E_{\theta\theta} = -\frac{1}{2}e^{2R} \left( 2VL^2 + \dot{A}^2 + 3\dot{R}^2 + 2\ddot{R} \right) \\
&\quad + \frac{e^{2R}}{32m^2L^2} \left[ \dot{A}^4 + 8\dot{R}\dot{A}^3 + \dot{A}^2 \left( 8\ddot{A} - 54\dot{R}^2 - 28\ddot{R} \right) - 67\dot{R}^4 \right. \\
&\quad + 4\dot{R}^2 \left( 44\ddot{A} - 61\ddot{R} \right) + 8\dot{A} \left( 16\dot{R}^3 - 13 \left( \ddot{A} - 2\ddot{R} \right) \dot{R} - 3\ddot{\ddot{A}} + 5\ddot{\ddot{R}} \right) \\
&\quad \left. - 8 \left( 3 \left( \ddot{A} - 3\ddot{R} \right) \left( \ddot{A} - \dot{R} \right) - 2\ddot{\ddot{A}} + 2\ddot{\ddot{R}} \right) + 8\dot{R} (11\ddot{\ddot{A}} - 13\ddot{\ddot{R}}) \right], \\
0 &= E_{t\theta} = -\frac{1}{2}e^{A+R} \left( 2VL^2 + \left( \dot{A} + \dot{R} \right)^2 + \ddot{A} + \ddot{R} \right) \\
&\quad + \frac{e^{A+R}}{32m^2L^2} \left( \dot{A} + \dot{R} \right)^2 \left[ \left( \dot{A} + \dot{R} \right)^2 + 2 \left( \ddot{A} + \ddot{R} \right) \right],
\end{aligned}$$

where  $\dot{\phantom{x}}$  denotes differentiation with respect to the new radial coordinate  $\eta$ .

## B Perturbative calculation in the original EOM

In this section, we perform some perturbative calculations in the original EOM, which reveal that full EOM need to be organized in terms of a certain formal expansion parameter  $\epsilon$ . This formal expansion parameter  $\epsilon$  is just the book-keeping one to retain the correct order in EOM and obtain consistent results, which should be set to be unity at the end of calculation. Only the relevant orders of  $\epsilon$  will be represented in the expansion expressions below. Furthermore, we do not use the trick exchanging the role of the radial coordinate and the scalar field  $\phi$ . All the variable are expanded in terms of the radial coordinate  $r$ . Therefore, this calculation may be regarded as the independent check of our results given in the main text. In the following, we will use  $(t, r, \theta)$  coordinate to show our results.

At the asymptotic of extremal black holes, we can consider following expansion of various variables in NMG as

$$\begin{aligned}
A(r) &= \ln r + \epsilon^2 \tilde{a}_1 r^{-2} + \dots, & B(r) &= -\ln r + \epsilon^2 \tilde{b}_1 r^{-2} + \dots, \\
\phi(r) &= \phi_\infty + \epsilon \frac{\tilde{\phi}_1}{r} + \dots, & V(\phi) &= -\frac{2}{L^2} \left( 1 - \frac{1}{4m^2 L^2} \right) - \frac{1}{2L^2} (\phi - \phi_\infty)^2 + \dots,
\end{aligned} \tag{C.1}$$

With the above expansion and the parameter  $\epsilon$ , the expressions for the scalar field equation and the metric EOM in NMG are obtained as

$$\begin{aligned}
E_\phi &= \mathcal{O}(\epsilon^3) + \dots, & E_{rr} &= \mathcal{O}(\epsilon^4) + \dots, \\
E_{\theta\theta} &= \frac{1}{2} \left[ 4q(\tilde{a}_1 + \tilde{b}_1) + \tilde{\phi}_1^2 \right] \epsilon^2 + \mathcal{O}(\epsilon^4) + \dots, \\
E_{t\theta} &= \frac{1}{2} \left[ 4q(\tilde{a}_1 + \tilde{b}_1) + \tilde{\phi}_1^2 \right] \epsilon^2 + \mathcal{O}(\epsilon^4) + \dots,
\end{aligned}$$

where  $q \equiv 1 - 1/2m^2 L^2$ .

Through  $0 = E_\phi$  and  $0 = E_{\mu\nu}$ , we obtain the following relation

$$q(\tilde{a}_1 + \tilde{b}_1) = -\frac{1}{4}\tilde{\phi}_1^2, \quad (\text{C.2})$$

which is consistent with the results from reduced EOM in NMG. However, as we have alluded to in the main text, the reduced EOM can be integrated partially with the integration constant  $\tilde{\Delta}_0$ , the constants  $\tilde{a}_1$  and  $\tilde{b}_1$  can be determined completely in terms of  $\tilde{\Delta}_0$  and  $\tilde{\omega}_1$ , while that is not possible in the perturbative calculation in the full EOM.

On the near horizon, we have the expansion of various variables as

$$\begin{aligned} A(r) &= \ln a(r - r_H) + \dots, & B(r) &= -\ln b(r - r_H) + \dots, \\ \phi(r) &= \phi_H + \frac{\epsilon}{h_0}(r - r_H) + \dots, \\ V(\phi) &= -\frac{\mathcal{W}(\phi_H)^2}{2L^2} \left[ \left( 1 - \frac{\mathcal{W}(\phi_H)^2}{16m^2L^2} \right) - 2(\phi - \phi_H)^2 \right] + \dots. \end{aligned} \quad (\text{C.3})$$

With the above expansion, the expressions for scalar field equation and the metric EOM can be written as

$$\begin{aligned} E_\phi &= \mathcal{O}(\epsilon) + \dots, \\ E_{rr} &= \frac{1}{32b^2m^2L^2} \left( b^2 - \mathcal{W}(\phi_H)^2 \right) \left( b^2 - 16m^2L^2 + \mathcal{W}(\phi_H)^2 \right) (r - r_H)^{-2} + \mathcal{O}(\epsilon^2) + \dots, \\ E_{\theta\theta} &= \frac{r_H^2}{32m^2L^2} \left( b^2 - \mathcal{W}(\phi_H)^2 \right) \left( b^2 - 16m^2L^2 + \mathcal{W}(\phi_H)^2 \right) + \mathcal{O}(\epsilon^2) + \dots, \\ E_{t\theta} &= \frac{a r_H}{32m^2L^2} \left( b^2 - \mathcal{W}(\phi_H)^2 \right) \left( b^2 - 16m^2L^2 + \mathcal{W}(\phi_H)^2 \right) (r - r_H) + \mathcal{O}(\epsilon^2) + \dots. \end{aligned}$$

From  $0 = E_\phi$  and  $0 = E_{\mu\nu}$ , we obtain

$$b^2 = \mathcal{W}(\phi_H)^2, \quad (\text{C.4})$$

which is also consistent with the results obtained from the reduced EOM.

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